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<https://doi.org/10.1090/conm/631/12605>

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# A note on three problems in metric Diophantine approximation

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November 2, 2013

*Dedicated to Shrikrishna Gopalrao Dani  
on the occasion of his 65th birthday*

## Abstract

The use of Hausdorff measures and dimension in the theory of Diophantine approximation dates back to the 1920s with the theorems of Jarník and Besicovitch regarding well-approximable and badly-approximable points. In this paper we consider three inhomogeneous problems that further develop these classical results. Firstly, we obtain a Jarník type theorem for the set  $\mathcal{S}_2^\times(\psi; \boldsymbol{\theta})$  of multiplicatively approximable points in the plane  $\mathbb{R}^2$ . This Hausdorff measure statement does not reduce to Gallagher's Lebesgue measure statement as one might expect and is new even in the homogeneous setting ( $\boldsymbol{\theta} = \mathbf{0}$ ). Next, we establish a Jarník type theorem for the set  $\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathcal{C}$  where  $\mathcal{C}$  is a non-degenerate planar curve. This completes the Hausdorff theory for planar curves and clarifies a potential oversight in [2]. Finally, we show that the set  $\mathbf{Bad}(i, j; \boldsymbol{\theta})$  of simultaneously inhomogeneously badly approximable points in  $\mathbb{R}^2$  is of full dimension. The underlying philosophy behind the proof has other applications; e.g. towards establishing the inhomogeneous version of Schmidt's Conjecture. The higher dimensional analogues of the planar results are also discussed.

2000 *Mathematics Subject Classification*: Primary 11J83; Secondary 11J13, 11K60

*Keywords and phrases*: Metric Diophantine approximation, Jarník type theorems, Hausdorff dimension and measure, multiplicative and inhomogeneous simultaneous approximation.

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\*Research supported by EPSRC grant EP/J018260/1

†Research supported by EPSRC grants EP/E061613/1, EP/F027028/1 and EP/J018260/1

# 1 Multiplicatively $\psi$ -well approximable points

Throughout  $\psi : \mathbb{N} \rightarrow [0, +\infty)$  is a non-negative function. We will normally assume that  $\psi$  is strictly positive and monotonically decreasing in which case  $\psi$  will be referred to as an *approximating function*. Given  $\psi$ , a real number  $x$  will be called  *$\psi$ -well approximable* or simply  *$\psi$ -approximable* if there are infinitely many  $q \in \mathbb{N}$  such that

$$\|qx\| < \psi(q).$$

Here and throughout  $\|\cdot\|$  denotes the distance of a real number to the nearest integer. Let  $\mathcal{S}_1(\psi)$  denote the set of all  $\psi$ -approximable real numbers. The set  $\mathcal{S}_1(\psi)$  is invariant under translations by integers. Hence, we will often restrict  $x$  to lie in the unit interval  $\mathbb{I} := [0, 1]$ .

The well known theorem of Dirichlet states that  $\mathcal{S}_1(\psi) = \mathbb{R}$  when  $\psi(q) = q^{-1}$ . In turn, a rather simple consequence of the Borel-Cantelli lemma from probability theory is that  $\mathcal{S}_1(\psi)$  is null (that is of Lebesgue measure zero) whenever  $\sum_{q=1}^{\infty} \psi(q) < \infty$ . However, Khintchine's theorem [24] tells us that the set  $\mathcal{S}_1(\psi)$  is full (that is its complement is of Lebesgue measure zero) whenever  $\sum_{q=1}^{\infty} \psi(q) = \infty$  and  $\psi$  is monotonic. In order to quantify the size of  $\mathcal{S}_1(\psi)$  when it is null, Jarník [22] and Besicovitch [14] pioneered the use of Hausdorff measures and dimension. Throughout,  $\dim X$  will denote the Hausdorff dimension of a subset  $X$  of  $\mathbb{R}^n$  and  $\mathcal{H}^s(X)$  the  $s$ -dimensional Hausdorff measure (see §1.1.2 for the definition and further details). The modern version of the classical Jarník-Besicovitch theorem (see [7] or [8]) states that for any approximating function  $\psi$

$$\dim \mathcal{S}_1(\psi) = \min \left\{ 1, \frac{2}{\tau + 1} \right\} \quad \text{where} \quad \tau := \liminf_{q \rightarrow \infty} \frac{-\log \psi(q)}{\log q}. \quad (1)$$

In other words, the ‘modern theorem’ relates the Hausdorff dimension of  $\mathcal{S}_1(\psi)$  to the lower order at infinity of  $1/\psi$  and up to a certain degree allows us to discriminate between  $\psi$ -approximable sets of Lebesgue measure zero. A more delicate measurement of the ‘size’ of  $\mathcal{S}_1(\psi)$  is obtained by expressing the size in terms of Hausdorff measures  $\mathcal{H}^s$ . With respect to such measures, the modern version of Jarník theorem (see [7] or [8]) states that for any  $s \in (0, 1)$  and any approximating function  $\psi$

$$\mathcal{H}^s(\mathcal{S}_1(\psi) \cap \mathbb{I}) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{1-s} \psi^s(q) < \infty, \\ \mathcal{H}^s(\mathbb{I}) & \text{if } \sum_{q=1}^{\infty} q^{1-s} \psi^s(q) = \infty. \end{cases} \quad (2)$$

Note that for  $0 < s < 1$  we have that  $\mathcal{H}^s(\mathbb{I}) = \infty$ . However, since  $\mathcal{H}^1(\mathbb{I}) = 1$ , the statement as written also holds for  $s = 1$  due to the aforementioned theorem of Khintchine. Note that it is trivially true for  $s > 1$ . The upshot is that statement (2) is true for any  $s > 0$  and is referred to as the *Khintchine-Jarník theorem*. It is worth pointing out that there is an even more general version of (2) that makes use of more general Hausdorff measures, see [7, 8, 10, 17]. Within this paper we restrict ourselves to the case of  $s$ -dimensional Hausdorff measures.

In higher dimensions there are various natural generalizations of  $\mathcal{S}_1(\psi)$ . Given an approximating function  $\psi$ , the point  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  will be called  *$\psi$ -well approximable* or simply  *$\psi$ -approximable* if there are infinitely many  $q \in \mathbb{N}$  such that

$$\max\{\|qx_1\|, \dots, \|qx_n\|\} < \psi(q) \quad (3)$$

and it will be called *multiplicatively  $\psi$ -well approximable* or simply *multiplicatively  $\psi$ -approximable* if there are infinitely many  $q \in \mathbb{N}$  such that

$$\|qx_1\| \cdots \|qx_n\| < \psi(q). \quad (4)$$

Denote by  $\mathcal{S}_n(\psi)$  the set of  $\psi$ -approximable points in  $\mathbb{R}^n$  and by  $\mathcal{S}_n^\times(\psi)$  the set of multiplicatively  $\psi$ -approximable points in  $\mathbb{R}^n$ . On comparing (3) and (4) one easily spots that

$$\mathcal{S}_n(\psi^{1/n}) \subset \mathcal{S}_n^\times(\psi).$$

For the sake of clarity, in what follows we will mainly restrict our attention to the case of the plane  $\mathbb{R}^2$ . The Khintchine-Jarník theorem for  $\mathcal{S}_2(\psi)$  (see [7] or [8]) states that for any  $s > 0$  and any approximating function  $\psi$

$$\mathcal{H}^s(\mathcal{S}_2(\psi) \cap \mathbb{I}^2) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{2-s} \psi^s(q) < \infty, \\ \mathcal{H}^s(\mathbb{I}^2) & \text{if } \sum_{q=1}^{\infty} q^{2-s} \psi^s(q) = \infty. \end{cases} \quad (5)$$

Regarding the Lebesgue case, which corresponds to when  $s = 2$ , Gallagher [21] showed that the monotonicity of  $\psi$  is unnecessary. As a consequence of the Mass Transference Principle [10] we have that (5) holds for any  $\psi$  (not necessarily monotonic) and any  $s > 0$ .

In the multiplicative setup, Gallagher [20] essentially proved that for any approximating function  $\psi$

$$\mathcal{H}^2(\mathcal{S}_2^\times(\psi) \cap \mathbb{I}^2) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) \log q < \infty, \\ \mathcal{H}^2(\mathbb{I}^2) & \text{if } \sum_{q=1}^{\infty} \psi(q) \log q = \infty. \end{cases} \quad (6)$$

The extra log factor in the above sum accounts for the larger volume of the fundamental domains defined by (4) compared to (3). The recent work [9] has made an attempt to relax the monotonicity assumption on  $\psi$  within the multiplicative setting. Our goal in this paper is to investigate the Hausdorff measure theory within the multiplicative setting.

**Problem 1:** *Determine the Hausdorff measure  $\mathcal{H}^s$  of  $\mathcal{S}_n^\times(\psi)$ .*

This problem is somewhat different to the non-multiplicative setting where we have the uniform solution given by (5). First of all, we note that

$$\text{if } s \leq 1 \text{ then } \mathcal{H}^s(\mathcal{S}_2^\times(\psi) \cap \mathbb{I}^2) = \infty \text{ irrespective of approximating function } \psi. \quad (7)$$

To see this, we observe that for any  $\psi$ -approximable number  $\alpha \in \mathbb{R}$  the whole line  $x_1 = \alpha$  is contained in  $\mathcal{S}_2^\times(\psi)$ . Hence,

$$\mathcal{S}_1(\psi) \times \mathbb{R} \subset \mathcal{S}_2^\times(\psi). \quad (8)$$

It is easy to verify (for example, by using the theory of continued fractions) that  $\mathcal{S}_1(\psi)$  is an infinite set for any approximating function  $\psi$  and so (8) implies (7). Next, since  $\mathcal{S}_2^\times(\psi) \subseteq \mathbb{R}^2$ , we trivially have that

$$\text{if } s > 2 \text{ then } \mathcal{H}^s(\mathcal{S}_2^\times(\psi) \cap \mathbb{I}^2) = 0 \text{ irrespective of } \psi.$$

The upshot of this and (7) is that when attacking Problem 1, there is no loss of generality in assuming that  $s \in (1, 2]$ . Furthermore, the Lebesgue case ( $s = 2$ ) is covered by Gallagher's result so we may as well assume that  $1 < s < 2$ .

Recall that Gallagher's multiplicative statement (6) has the extra 'log factor' in the 'volume' sum compared to the simultaneous statement (5). It is natural to expect the log factor to appear in one form or another when determining the Hausdorff measure  $\mathcal{H}^s$  of  $\mathcal{S}_2^\times(\psi)$  for  $s \in (1, 2)$ ; in other words when  $s$  is not an integer and so  $\mathcal{H}^s$  is genuinely a fractal measure. This, as we shall soon see, is very far from the truth. The 'log factor' completely disappears! Thus, genuine 'fractal' Hausdorff measures are insensitive to the multiplicative nature of  $\mathcal{S}_2^\times(\psi)$ . Indeed, what we essentially have is that

$$\mathcal{H}^s(\mathcal{S}_2^\times(\psi)) = \mathcal{H}^{s-1}(\mathcal{S}_1(\psi)).$$

Thus, that for  $s < 1$  the  $s$ -dimensional Hausdorff measure of both sides of (8) is the same. In short, for any  $s \in (0, 1)$ , the points of  $\mathcal{S}_2^\times(\psi)$  that do not lie in  $\mathbb{R} \times \mathcal{S}_1(\psi)$  do not contribute any substantial 'mass' in terms of the associated  $s$ -dimensional Hausdorff measure.

The ideas and tricks used in our investigation of Problem 1 are equally valid within the more general inhomogeneous setup: given an approximating function  $\psi$  and a fixed point  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ , let  $\mathcal{S}_n^\times(\psi; \boldsymbol{\theta})$  denote the set of points  $(x_1, \dots, x_n) \in \mathbb{R}^n$  such that there are infinitely many  $q \in \mathbb{N}$  satisfying the inequality

$$\|qx_1 - \theta_1\| \cdots \|qx_n - \theta_n\| < \psi(q). \quad (9)$$

We prove the following inhomogeneous statement.

**Theorem 1** *Let  $\psi$  be an approximating function,  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$  and  $s \in (1, 2)$ . Then*

$$\mathcal{H}^s(\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{2-s} \psi^{s-1}(q) < \infty, \\ \mathcal{H}^s(\mathbb{I}^2) & \text{if } \sum_{q=1}^{\infty} q^{2-s} \psi^{s-1}(q) = \infty. \end{cases} \quad (10)$$

*Remark 1.1.* Note that  $\mathcal{H}^s(\mathbb{I}^2) = \infty$  when  $s < 2$ . We reiterate the fact that unlike the Khintchine-Jarník theorem, the statement of Theorem 1 is false when  $s = 2$ .

*Remark 1.2.* In higher dimensions, Gallagher's multiplicative statement reads

$$\mathcal{H}^n(\mathcal{S}_n^\times(\psi) \cap \mathbb{I}^2) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) \log^{n-1} q < \infty, \\ \mathcal{H}^2(\mathbb{I}^2) & \text{if } \sum_{q=1}^{\infty} \psi(q) \log^{n-1} q = \infty. \end{cases}$$

For  $n > 2$ , the proof of Theorem 1 can be adapted to show that for any  $s \in (n-1, n)$

$$\mathcal{H}^s(\mathcal{S}_n^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^n) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} q^{n-s} \psi^{s+1-n}(q) \log^{n-2} q < \infty.$$

Thus, for convergence in higher dimensions we lose a log factor from the Lebesgue volume sum appearing in Gallagher's result. This of course is absolutely consistent with the  $n = 2$  situation given by Theorem 1. Regarding a divergent statement, the arguments used in proving Theorem 1 can be adapted to show that for any  $s \in (n-1, n)$

$$\mathcal{H}^s(\mathcal{S}_n^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^n) = \mathcal{H}^s(\mathbb{I}^n) \quad \text{if} \quad \sum_{q=1}^{\infty} q^{n-s} \psi^{s+1-n}(q) = \infty.$$

Thus, there is a discrepancy in the above 's-volume' sum conditions for convergence and divergence when  $n > 2$ . In view of this, it remains an interesting open problem to determine the necessary and sufficient condition for  $\mathcal{H}^s(\mathcal{S}_n^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^n)$  to be zero or infinite in higher dimensions.

## 1.1 Proof of Theorem 1

To simplify notation the symbols  $\ll$  and  $\gg$  will be used to indicate an inequality with an unspecified positive multiplicative constant. If  $a \ll b$  and  $a \gg b$  we write  $a \asymp b$ , and say that the quantities  $a$  and  $b$  are comparable. For a real number  $x$ , the quantity  $\{x\}$  will denote the fractional part of  $x$  and  $[x]$  the integer part of  $x$ .

Without loss of generality, throughout the proof of Theorem 1 we can assume that  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{I}^2$ .

### 1.1.1 A covering of $\mathcal{S}_2^\times(\psi, \boldsymbol{\theta}) \cap \mathbb{I}^2$

In this section we obtain an effective covering of the set  $\mathcal{S}_2^\times(\psi, \boldsymbol{\theta}) \cap \mathbb{I}^2$  that will be used in establishing the convergence case of Theorem 1.

**Lemma 1** Let  $0 < \varepsilon < 1$ ,  $(x_1, x_2) \in \mathbb{I}^2$ ,  $(\theta_1, \theta_2) \in \mathbb{I}^2$ ,  $q \in \mathbb{N}$  and

$$\prod_{i=1}^2 \|qx_i - \theta_i\| < \varepsilon. \quad (11)$$

Then there exist  $m \in \mathbb{Z}$  and  $p_1, p_2 \in \{-1, 0, \dots, q\}$  such that

$$\begin{aligned} \|qx_i - \theta_i\| &= |qx_i - \theta_i - p_i| \quad \text{for } i = 1, 2, \\ \|qx_1 - \theta_1\| &< 2^m \sqrt{2\varepsilon}, \quad \|qx_2 - \theta_2\| < 2^{-m} \sqrt{2\varepsilon} \end{aligned} \quad (12)$$

and

$$2^{|m|} \sqrt{\varepsilon} \leq 1. \quad (13)$$

PROOF. The existence of  $p_i \in \{-1, 0, \dots, q\}$  with  $|qx_i - p_i - \theta_i| = \|qx_i - \theta_i\|$  is an immediate consequence of the fact that  $x_i, \theta_i \in \mathbb{I}$ . Thus, the only thing that we need to prove is the existence of  $m$  satisfying (12) and (13).

If  $\|qx_i - \theta_i\| < \sqrt{2\varepsilon}$  for each  $i = 1, 2$  then we can define  $m = 0$ . In this case (12) is obvious and (13) is a consequence of the fact that  $0 < \varepsilon < 1$ .

Without loss of generality, assume that  $\|qx_1 - \theta_1\| \geq \sqrt{2\varepsilon}$  and let  $m \in \mathbb{Z}$  be the unique integer such that

$$2^{m-1} \sqrt{2\varepsilon} \leq \|qx_1 - \theta_1\| < 2^m \sqrt{2\varepsilon}.$$

Since  $\|qx_1 - \theta_1\| \geq \sqrt{2\varepsilon}$ , we have that  $m \geq 0$ . Furthermore, since  $\|qx_1 - \theta_1\| \leq 1/2$ , we have that  $2^m \sqrt{\varepsilon} < 1$  whence (13) follows. The left hand side of (12) holds by the definition of  $m$ . To show the right hand side of (13) we use (11). Indeed, we have that

$$2^{m-1} \sqrt{2\varepsilon} \|qx_2 - \theta_2\| \leq \prod_{i=1}^2 \|qx_i - \theta_i\| < \varepsilon$$

whence the right hand side of (13) follows. This completes the proof of the lemma.  $\square$

**Lemma 2** Let  $\psi : \mathbb{N} \rightarrow [0, 1)$  be decreasing and  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{I}^2$ . Then for any  $\ell \in \mathbb{N}$

$$\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2 \subset \bigcup_{t=\ell}^{\infty} \bigcup_{2^t \leq q < 2^{t+1}} \bigcup_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} \bigcup_{p_1=-1}^q \bigcup_{p_2=-1}^q S_{\boldsymbol{\theta}}(t, q, m, p_1, p_2), \quad (14)$$

where

$$S_{\boldsymbol{\theta}}(t, q, m, p_1, p_2) := \left\{ (x_1, x_2) \in \mathbb{I}^2 : \begin{aligned} &\left| x_1 - \frac{p_1 + \theta_1}{q} \right| < \frac{2^m \sqrt{2\psi(2^t)}}{2^t} \\ &\left| x_2 - \frac{p_2 + \theta_2}{q} \right| < \frac{2^{-m} \sqrt{2\psi(2^t)}}{2^t} \end{aligned} \right\}.$$

PROOF. It is easily verified that

$$\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2 = \bigcap_{\ell=1}^{\infty} \bigcup_{t=\ell}^{\infty} \bigcup_{2^t \leq q < 2^{t+1}} \left\{ (x_1, x_2) \in \mathbb{I}^2 : \prod_{i=1}^2 \|qx_i - \theta_i\| < \psi(q) \right\}.$$

Since  $\psi$  is decreasing,  $\psi(q) \leq \psi(2^t)$  for  $2^t \leq q < 2^{t+1}$ . Then, for any  $\ell \in \mathbb{N}$

$$\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2 \subset \bigcup_{t=\ell}^{\infty} \bigcup_{2^t \leq q < 2^{t+1}} \left\{ (x_1, x_2) \in \mathbb{I}^2 : \prod_{i=1}^2 \|qx_i - \theta_i\| < \psi(2^t) \right\}. \quad (15)$$

For a fixed pair  $t$  and  $q$  with  $2^t \leq q < 2^{t+1}$ , by Lemma 1 with  $\varepsilon = \psi(2^t)$ , we get that

$$\left\{ (x_1, x_2) \in \mathbb{I}^2 : \prod_{i=1}^2 \|qx_i - \theta_i\| < \psi(2^t) \right\} \subset \bigcup_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} \bigcup_{p_1=-1}^q \bigcup_{p_2=-1}^q S_{\boldsymbol{\theta}}(t, q, m, p_1, p_2).$$

This together with (15) completes the proof of the lemma.  $\square$

### 1.1.2 Hausdorff measure and dimension

We briefly recall various facts regarding Hausdorff measures that will be used in the course of establishing Theorem 1. Given  $\delta > 0$  and a set  $X \subset \mathbb{R}^n$ , any finite or countable collection  $\{B_i\}$  of subsets of  $\mathbb{R}^n$  such that

$$X \subset \bigcup_i B_i \quad (\text{i.e. } \{B_i\} \text{ is a cover for } X)$$

and

$$\text{diam } B_i \leq \delta \quad \text{for all } i$$

is called a  $\delta$ -cover of  $X$ . Given a real number  $s$ , let

$$\mathcal{H}_\delta^s(X) := \inf_{\{B_i\}} \sum_i \text{diam}(B_i)^s,$$

where the infimum is taken over all possible  $\delta$ -covers  $\{B_i\}$  of  $X$ . The  $s$ -dimensional Hausdorff measure  $\mathcal{H}^s(X)$  of  $X$  is defined to be

$$\mathcal{H}^s(X) := \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^s(X)$$

and the Hausdorff dimension  $\dim X$  of  $X$  by

$$\dim X := \inf\{s : \mathcal{H}^s(X) = 0\} = \sup\{s : \mathcal{H}^s(X) = \infty\}.$$

The countable collection  $\{B_i\}$  is called a *fine cover* of  $X$  if for every  $\delta > 0$  it contains a subcollection that is a  $\delta$ -cover of  $X$ . The following statement is an immediate and well known consequence of the definition of  $\mathcal{H}^s$ .



**Lemma 3** *Let  $\{B_i\}$  be a fine cover of  $X$  and  $s > 0$  be such that  $\sum_i \text{diam}(B_i)^s < \infty$ . Then*

$$\mathcal{H}^s(X) = 0.$$

### 1.1.3 Proof: the convergence case

We are given that  $\sum_{q=1}^{\infty} q^{2-s} \psi(q)^{s-1} < \infty$ . As already mentioned, we can assume that  $\boldsymbol{\theta} \in \mathbb{I}^2$ . The proof will make use of the covering of  $\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2$  given by Lemma 2. The rectangle  $S_{\boldsymbol{\theta}}(t, q, m, p_1, p_2)$  arising from this lemma has sides of lengths

$$A := \frac{2^{-|m|+1} \sqrt{2\psi(2^t)}}{2^t} \quad \text{and} \quad B := \frac{2^{|m|+1} \sqrt{2\psi(2^t)}}{2^t}$$

and so can be split into  $B/A = 2^{2|m|}$  squares with sidelength  $A$ . By Lemma 2, the collection of such squares taken over  $t \geq \ell$  and over  $q, p_1, p_2, m$  as specified in the lemma is a  $\delta$ -cover of  $\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2$  with  $\delta := \sqrt{2}A \rightarrow 0$  as  $\ell \rightarrow \infty$ . Therefore, the collection of all such squares, say  $\{B_i\}$ , is a fine cover of  $\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2$ . It follows that

$$\begin{aligned} \sum_i \text{diam}(B_i)^s &\ll \sum_{t=\ell}^{\infty} \sum_{2^t \leq q < 2^{t+1}} \sum_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} \sum_{p_1=-1}^q \sum_{p_2=-1}^q \left( 2^{-|m|} \frac{\sqrt{\psi(2^t)}}{2^t} \right)^s 2^{2|m|} \\ &\ll \sum_{t=\ell}^{\infty} \sum_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} \left( 2^{-|m|} \frac{\sqrt{\psi(2^t)}}{2^t} \right)^s 2^{2|m|} 2^{3t} \\ &\ll \sum_{t=\ell}^{\infty} \left( \frac{\sqrt{\psi(2^t)}}{2^t} \right)^s 2^{3t} \sum_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} 2^{(2-s)|m|}. \end{aligned} \quad (16)$$

Since  $1 < s < 2$ , the sum over  $m \neq 0$  in the right hand side of (16) is a finite increasing geometric progression, which is easily estimated to give

$$\sum_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} 2^{(2-s)|m|} \ll \left( \frac{1}{\sqrt{\psi(2^t)}} \right)^{2-s} = (\sqrt{\psi(2^t)})^{s-2}. \quad (17)$$

Substituting this into (16) gives

$$\begin{aligned} \sum_i \text{diam}(B_i)^s &\ll \sum_{t=\ell}^{\infty} \left( \frac{\sqrt{\psi(2^t)}}{2^t} \right)^s 2^{3t} (\sqrt{\psi(2^t)})^{s-2} \\ &= \sum_{t=\ell}^{\infty} 2^{(3-s)t} \psi(2^t)^{s-1} \ll \sum_{q=1}^{\infty} q^{2-s} \psi(q)^{s-1} < \infty. \end{aligned}$$

By Lemma 3,  $\mathcal{H}^s(\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2) = 0$  and thus the proof of the convergence part is complete.

### 1.1.4 Proof: the divergence case

We are given that  $\sum_{q=1}^{\infty} q^{2-s} \psi(q)^{s-1} = \infty$ . Then, by the inhomogeneous version of Jarník's theorem [15] (see also the remark in [8, §12.1]), it follows that  $\mathcal{H}^{s-1}(\mathcal{S}_1(\psi; \theta_1) \cap \mathbb{I}) = \infty$ . The observation that led to (8) is equally valid in the inhomogeneous setup; that is to say that

$$\mathcal{S}_1(\psi; \theta_1) \times \mathbb{R} \subset \mathcal{S}_2^{\times}(\psi; \boldsymbol{\theta}).$$

Thus,  $\mathcal{H}^s(\mathcal{S}_2^{\times}(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2) \geq \mathcal{H}^s((\mathcal{S}_1(\psi; \theta_1) \cap \mathbb{I}) \times \mathbb{I})$ . Since  $\mathcal{H}^{s-1}(\mathcal{S}_1(\psi; \theta_1) \cap \mathbb{I}) = \infty$ , the slicing lemma [11, Lemma 4] implies that

$$\mathcal{H}^s((\mathcal{S}_1(\psi; \theta_1) \cap \mathbb{I}) \times \mathbb{I}) = \infty.$$

Hence  $\mathcal{H}^s(\mathcal{S}_2^{\times}(\psi; \boldsymbol{\theta}) \cap \mathbb{I}^2) = \infty$  and the proof of Theorem 1 is complete.

## 2 Diophantine approximation on planar curves

When the coordinates of the approximated point  $\mathbf{x} \in \mathbb{R}^n$  are confined by functional relations, we fall into the theory of Diophantine approximation on manifolds [13]. Over the last decade or so, the theory of Diophantine approximation on manifolds has developed at some considerable pace with the catalyst being the pioneering work of Kleinbock & Margulis [25]. For details of this and an overview of the almost complete results regarding  $\mathcal{S}_n(\psi)$  restricted to manifolds  $\mathcal{M} \subset \mathbb{R}^n$  see [6, 8] and references within. However, much less is known regarding multiplicative Diophantine approximation on manifolds. It would be highly desirable to address this imbalance by investigating the following analogue of Problem 1 for manifolds.

**Problem 2:** *Determine the Hausdorff measure  $\mathcal{H}^s$  of  $\mathcal{S}_n^{\times}(\psi) \cap \mathcal{M}$ .*

Our goal in this paper is to consider the problem in the case  $\mathcal{M}$  is a planar curve  $\mathcal{C}$  and  $\mathcal{H}^s$  is a genuine fractal measure.

**Theorem 2** *Let  $\psi$  be any approximating function,  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$  and  $s \in (0, 1)$ . Let  $\mathcal{C}$  be a  $C^{(3)}$  curve in  $\mathbb{R}^2$  with non-zero curvature everywhere apart from a set of  $s$ -dimensional Hausdorff measure zero. Then*

$$\mathcal{H}^s(\mathcal{S}_2^{\times}(\psi; \boldsymbol{\theta}) \cap \mathcal{C}) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{1-s} \psi^s(q) < \infty, \\ \infty & \text{if } \sum_{q=1}^{\infty} q^{1-s} \psi^s(q) = \infty. \end{cases} \quad (18)$$

*Remark 2.1.* In [2], the authors prove that  $\mathcal{H}^s(\mathcal{S}_2^{\times}(\psi) \cap \mathcal{C}) = 0$  under the more restrictive assumption that  $\sum_{q=1}^{\infty} q^{1-s} \psi^s(q) (\log q)^s < \infty$ . Although not an error, in their proof of

this homogeneous statement [2, Theorem 1] there is a certain degree of ambiguity in the manner in which a key counting estimate originating from [30, Theorem 1] is applied. More precisely, it is important to stress that the implied constant appearing in inequality (13) associated with [2, Theorem VV] is independent of  $\psi$ . This is crucial as it is applied over a countable family of functions  $\psi(Q)$  that depend on a parameter  $m \in \mathbb{Z}$ .

## 2.1 Proof of Theorem 2

### 2.1.1 Rational points near planar curves

The proof of Theorem 2 relies on the results obtained in [12] regarding the distribution of ‘shifted’ rational points near planar curves, which we recall here. In view of the metrical nature of Theorem 2, there is no loss of generality in assuming that  $\mathcal{C} := \{(x, f(x)) : x \in I\}$  is the graph of a  $C^{(3)}$  function  $f : I \rightarrow \mathbb{R}$  defined on a finite closed interval  $I$  and that  $f''$  is continuous and non-vanishing on  $I$ . By the compactness of  $I$ , there exist positive and finite constants  $c_1, c_2$  such that

$$c_1 \leq |f''(x)| \leq c_2 \quad \text{for all } x \in I. \quad (19)$$

Given  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ ,  $\delta > 0$  and  $Q \geq 1$ , consider the set

$$A_{\boldsymbol{\theta}}(Q, \delta) := \left\{ (p_1, q) \in \mathbb{Z} \times \mathbb{N} : \begin{array}{l} Q < q \leq 2Q, (p_1 + \theta_1)/q \in I \\ \|qf((p_1 + \theta_1)/q) - \theta_2\| < \delta \end{array} \right\}.$$

The function  $N_{\boldsymbol{\theta}}(Q, \delta) = \#A_{\boldsymbol{\theta}}(Q, \delta)$  counts the number of rational points  $(p_1/q, p_2/q)$  with bounded denominator  $q$  such that the shifted points  $((p_1 + \theta_1)/q, (p_2 + \theta_2)/q)$  lie within the  $\delta/Q$ -neighborhood of the curve  $\mathcal{C}$ . The following result is a direct consequence of Theorem 3 from [12] – the inhomogeneous generalization of Theorem 1 from [30].

**Theorem 3** *Let  $f \in C^{(3)}(I)$  and satisfy (19) and let  $\varepsilon > 0$ . Then, for any  $Q \geq 1$  and  $0 < \delta \leq \frac{1}{2}$  we have that*

$$N_{\boldsymbol{\theta}}(Q, \delta) \ll \delta Q^2 + Q^{1+\varepsilon}$$

*where the implied constant is independent of  $Q$  and  $\delta$ .*

### 2.1.2 Proof: the convergence case

We are given that  $\sum_{q=1}^{\infty} q^{1-s} \psi(q)^s < \infty$ . Since  $\psi$  is monotonic, we have that

$$\sum_{t=1}^{\infty} 2^{(2-s)t} \psi(2^t)^s < \infty. \quad (20)$$

Hence

$$2^{(2-s)t}\psi(2^t)^s < 1 \quad \text{for all sufficiently large } t. \quad (21)$$

As in the proof of Theorem 1, without loss of generality we assume that  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{I}^2$  and moreover that  $\mathcal{C} \subset \mathbb{I}^2$ . By Lemma 2, for any  $\ell \in \mathbb{N}$  we have that

$$\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathcal{C} \subset \bigcup_{t=\ell}^{\infty} \bigcup_{2^t \leq q < 2^{t+1}} \bigcup_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} \bigcup_{p_1=-1}^q \bigcup_{p_2=-1}^q \mathcal{C} \cap S_{\boldsymbol{\theta}}(t, q, m, p_1, p_2). \quad (22)$$

Using condition (19), it is easily verified that

$$\text{diam}(\mathcal{C} \cap S_{\boldsymbol{\theta}}(t, q, p_1, p_2, m)) \ll 2^{-|m|} \frac{\sqrt{\psi(2^t)}}{2^t} \quad (23)$$

and that whenever  $\mathcal{C} \cap S_{\boldsymbol{\theta}}(t, q, p_1, p_2, m) \neq \emptyset$  we have that  $(p_1, q) \in A_{\boldsymbol{\theta}}(Q, \delta)$  with

$$\delta \asymp 2^{|m|} \sqrt{\psi(2^t)} \quad \text{and} \quad Q = 2^t, \quad (24)$$

where  $t, q, p_1, p_2, m$  are constrained as in (22) and the implied constants in (23) and (24) depend on  $c_1$  and  $c_2$  that appear in (19). By (22), the collection of all such sets  $\mathcal{C} \cap S_{\boldsymbol{\theta}}(t, q, m, p_1, p_2)$  is a fine cover of  $\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathcal{C}$ . By Theorem 3 with  $\varepsilon := s/4$ , we have that

$$N_{\boldsymbol{\theta}}(Q, \delta) \ll 2^{|m|} \sqrt{\psi(2^t)} 2^{2t} + 2^{(1+s/4)t}$$

and so the  $s$ -dimensional volume of the above fine cover is

$$\begin{aligned} & \ll \sum_{t=1}^{\infty} \sum_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} \left( 2^{-|m|} \frac{\sqrt{\psi(2^t)}}{2^t} \right)^s \left( 2^{|m|} \frac{\sqrt{\psi(2^t)}}{2^t} 2^{3t} + 2^{(1+s/4)t} \right) \\ & \ll \sum_{t=1}^{\infty} \left( \frac{\sqrt{\psi(2^t)}}{2^t} \right)^{s+1} 2^{3t} \sum_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} 2^{(1-s)|m|} + \\ & \quad + \sum_{t=1}^{\infty} \sum_{\substack{m \in \mathbb{Z} \\ 2^{|m|} \sqrt{\psi(2^t)} \leq 1}} \left( 2^{-|m|} \frac{\sqrt{\psi(2^t)}}{2^t} \right)^s 2^{(1+s/4)t} \\ & \stackrel{(17)}{\ll} \sum_{t=1}^{\infty} \left( \frac{\sqrt{\psi(2^t)}}{2^t} \right)^{s+1} 2^{3t} (\sqrt{\psi(2^t)})^{s-1} + \\ & \quad + \sum_{t=1}^{\infty} \left( \frac{\sqrt{\psi(2^t)}}{2^t} \right)^s 2^{(1+s/4)t} \\ & \stackrel{(21)}{\ll} \sum_{t=1}^{\infty} 2^{(2-s)t} \psi(2^t)^s + \sum_{t=1}^{\infty} 2^{-ts/4} \stackrel{(20)}{<} \infty. \end{aligned}$$

By Lemma 3,  $\mathcal{H}^s(\mathcal{C} \cap \mathcal{S}_2^\times(\psi; \boldsymbol{\theta})) = 0$  and the proof of the convergence part of Theorem 2 is complete.

### 2.1.3 Proof: the divergence case

We are given that  $\sum_{q=1}^{\infty} q^{1-s} \psi(q)^s = \infty$ . Then, by the inhomogeneous version of Jarník's theorem [15] (see also the remark in [8, §12.1]), we have that  $\mathcal{H}^s(\mathcal{S}_1(\psi; \theta_1) \cap I) = \infty$ . The same observation that led to (8), gives rise to the following obvious inclusion

$$X := \{(x, f(x)) : x \in \mathcal{S}_1(\psi; \theta_1) \cap I\} \subset \mathcal{C} \cap \mathcal{S}_2^\times(\psi; \boldsymbol{\theta}).$$

Since  $f \in C^{(1)}$ , we have that  $f$  is locally bi-Lipshitz and thus the map  $x \mapsto (x, f(x))$  preserves  $s$ -dimensional Hausdorff measure. Therefore,

$$\mathcal{H}^s(\mathcal{S}_2^\times(\psi; \boldsymbol{\theta}) \cap \mathcal{C}) \geq \mathcal{H}^s(X) = \mathcal{H}^s(\mathcal{S}_1(\psi; \theta_1) \cap I) = \infty$$

and so completes the proof of Theorem 2.

## 3 Inhomogeneous badly approximable points

A real number  $x$  is said to be *badly approximable* if there exists a positive constant  $c(x)$  such that

$$\|qx\| > c(x) q^{-1} \quad \forall q \in \mathbb{N}.$$

Here and throughout  $\|\cdot\|$  denotes the distance of a real number to the nearest integer. It is well known that the set **Bad** of badly approximable numbers is of Lebesgue measure zero. However, a result of Jarník [23] states that

$$\dim \mathbf{Bad} = 1. \tag{25}$$

Thus, in terms of dimension the set of badly approximable numbers is maximal; it has the same dimension as the real line.

In higher dimensions there are various natural generalizations of **Bad**. Restricting our attention to the plane  $\mathbb{R}^2$ , given a pair of real numbers  $i$  and  $j$  such that

$$0 \leq i, j \leq 1 \quad \text{and} \quad i + j = 1, \tag{26}$$

a point  $(x_1, x_2) \in \mathbb{R}^2$  is said to be  $(i, j)$ -*badly approximable* if there exists a positive constant  $c(x_1, x_2)$  such that

$$\max\{\|qx_1\|^{1/i}, \|qx_2\|^{1/j}\} > c(x_1, x_2) q^{-1} \quad \forall q \in \mathbb{N}.$$

Denote by **Bad**( $i, j$ ) the set of  $(i, j)$ -badly approximable points in  $\mathbb{R}^2$ . If  $i = 0$ , then we use the convention that  $x^{1/i} := 0$  and so **Bad**( $0, 1$ ) is identified with  $\mathbb{R} \times \mathbf{Bad}$ . That

is,  $\mathbf{Bad}(0, 1)$  consists of points  $(x_1, x_2)$  with  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbf{Bad}$ . The roles of  $x_1$  and  $x_2$  are reversed if  $j = 0$ . It easily follows from classical results in the theory of metric Diophantine approximation that  $\mathbf{Bad}(i, j)$  is of (two-dimensional) Lebesgue measure zero. Building upon the work of Davenport [16], it is shown in [29] that

$$\dim \mathbf{Bad}(i, j) = 2. \quad (27)$$

For alternative proofs and various strengthenings see [3, 19, 26, 27, 28]. In particular, a consequence of the main result in [3] is that the intersection of any finite number of  $(i, j)$ -badly approximable sets is of full dimension. Obviously this implies that the intersection of any two such sets is non-empty and thus establishes a conjecture of Wolfgang Schmidt dating back to the eighties. Most recently, Jinpeng An [1] has shown that the set  $\mathbf{Bad}(i, j)$  is winning in the sense of Schmidt games and thus the intersection of any countable number of  $(i, j)$ -badly approximable sets is of full dimension.

The goal in this paper is to obtain the analogue of (27) within the inhomogeneous setup.

**Problem 3:** *Find an analogue of (27) for inhomogeneous approximation.*

For  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ , let  $\mathbf{Bad}(i, j; \boldsymbol{\theta})$  denote the set of points  $(x_1, x_2) \in \mathbb{R}^2$  such that

$$\max\{ \|qx_1 - \theta_1\|^{1/i}, \|qx_2 - \theta_2\|^{1/j} \} > c(x_1, x_2) q^{-1} \quad \forall q \in \mathbb{N}.$$

Naturally, given  $\theta \in \mathbb{R}$  the inhomogeneous generalisation of the one-dimensional set  $\mathbf{Bad}$  is the set

$$\mathbf{Bad}(\theta) := \{x \in \mathbb{R} : \exists c(x) > 0 \text{ so that } \|qx - \theta\| > c(x) q^{-1} \quad \forall q \in \mathbb{N}\}$$

and so, for example,  $\mathbf{Bad}(0, 1; \boldsymbol{\theta})$  is identified with  $\mathbb{R} \times \mathbf{Bad}(\theta)$ . It is straightforward to deduce that  $\mathbf{Bad}(i, j; \boldsymbol{\theta})$  is of measure zero from the inhomogeneous version of Khintchine's theorems with varying approximating functions in each co-ordinate. We will prove the following full dimension statement which represents the inhomogeneous analogue of (27).

**Theorem 4** *Let  $i, j$  satisfy (26) and  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ . Then*

$$\dim \mathbf{Bad}(i, j; \boldsymbol{\theta}) = 2.$$

The basic philosophy behind the proof is simple and is likely to be applicable to other situations where the goal is to generalize a known homogenous badly approximable statement to the inhomogeneous setting – see Remark 3.5 below. The key is to exploit the known homogeneous ‘intervals construction’ proof and use the power of subtraction; namely

$$(\text{homogeneous construction}) + (\boldsymbol{\theta} - \boldsymbol{\theta} = \mathbf{0}) \implies (\text{inhomogeneous statement}).$$

Before moving onto the proof of Theorem 4, several remarks are in order.

*Remark 3.1.* For  $i$  and  $j$  fixed, the proof can be easily modified to deduce that the intersection of any finite number of  $\mathbf{Bad}(i, j; \boldsymbol{\theta})$  sets is of full dimension. In fact, by making use of standard trickery (such as the argument that proves that the countable intersection of winning sets is winning) one can actually deduce that for any countable sequence  $\boldsymbol{\theta}_t \in \mathbb{R}^2$

$$\dim \left( \bigcap_{t=1}^{\infty} \mathbf{Bad}(i, j; \boldsymbol{\theta}_t) \right) = 2.$$

*Remark 3.2.* In another direction, the proof can be adapted to obtain the following more general form of Theorem 4 in which the inhomogeneous factor  $\boldsymbol{\theta}$  depends on  $(x_1, x_2)$ . More precisely, let  $\boldsymbol{\theta} = (\theta_1, \theta_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and let  $\mathbf{Bad}(i, j; \boldsymbol{\theta})$  denote the set of points  $(x_1, x_2)$  such that

$$\max\{ \|qx_1 - \theta_1(x_1, x_2)\|^{1/i}, \|qx_2 - \theta_2(x_1, x_2)\|^{1/j} \} > c(x_1, x_2) q^{-1} \quad \forall q \in \mathbb{N}.$$

Then, if  $\theta_1 = \theta_1(x_1)$  and  $\theta_2 = \theta_2(x_2)$  are Lipschitz functions of one variable, we have that

$$\dim \mathbf{Bad}(i, j; \boldsymbol{\theta}) = 2.$$

As an example, this statement implies that there is a set of  $(x_1, x_2) \in \mathbb{R}^2$  of Hausdorff dimension 2 such that

$$\max\{ \|qx_1 - x_1^2\|^{1/i}, \|qx_2 - x_2^3\|^{1/j} \} > c(x_1, x_2) q^{-1} \quad \forall q \in \mathbb{N}.$$

It is worth pointing out that in the case  $i = j$ , the statement is also true if  $\theta_1 = \theta_1(x_1, x_2)$  and  $\theta_2 = \theta_2(x_1, x_2)$  are Lipschitz functions of two variables.

*Remark 3.3.* There is no difficulty in establishing the higher dimension analogue of Theorem 4. For any  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$  and  $n$ -tuple of real numbers  $i_1, \dots, i_n \geq 0$  such that  $\sum i_r = 1$ , denote by  $\mathbf{Bad}(i_1, \dots, i_n; \boldsymbol{\theta})$  the set of points  $(x_1, \dots, x_n) \in \mathbb{R}^n$  for which there exists a positive constant  $c(x_1, \dots, x_n)$  such that

$$\max\{ \|qx_1 - \theta_1\|^{1/i_1}, \dots, \|qx_n - \theta_n\|^{1/i_n} \} > c(x_1, \dots, x_n) q^{-1} \quad \forall q \in \mathbb{N}.$$

By modifying the proof of Theorem 4, in the obvious way, it is easy to show that

$$\dim \mathbf{Bad}(i_1, \dots, i_n; \boldsymbol{\theta}) = n.$$

Moreover, the various proofs of the homogeneous results obtained in [28] regarding  $\mathbf{Bad}(i_1, \dots, i_n) \cap \Omega$ , where  $\Omega$  is some ‘nice’ fractal set (essentially, the support set of an absolutely friendly, Ahlfors regular measure) can be adapted to give the corresponding inhomogeneous statements without any serious difficulty. We have decided to restrict ourselves to proving Theorem 4 since it already contains the necessary ingredients to obtain the inhomogeneous statement from the homogeneous proof.

*Remark 3.4.* In the symmetric case  $i_1 = \dots = i_n = 1/n$ , our Theorem 4 and indeed its generalizations mentioned in the previous remark are covered by the work of Einsiedler &

Tseng [18, Theorem 1.1]. They actually deal with badly approximable systems of linear forms and show that the intersection of such sets with the support set  $\Omega$  of an absolutely friendly measure is winning in the sense of Schmidt games. We mention in passing, that Einsiedler & Tseng proved their results roughly at the same time as us, but for some mystical reason, it has taken us over four years to present our work. Indeed the second author had a useful discussion with Einsiedler regarding their preprint at the conference ‘The Diverse Faces of Arithmetic’ in honour of the late Graham Everest in 2009.

*Remark 3.5.* The basic philosophy behind the proof of Theorem 4 can be exploited to yield the inhomogeneous strengthening of Schmidt’s Conjecture. More precisely, we are able show that any inhomogeneous  $\mathbf{Bad}(i, j; \boldsymbol{\theta})$  set is winning and thus

$$\dim \left( \bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t; \boldsymbol{\theta}_t) \right) = 2.$$

Furthermore, it is possible to show that the intersection of  $\mathbf{Bad}(i, j; \boldsymbol{\theta})$  with any non-degenerate planar curve  $\mathcal{C}$  is winning as is the intersection with any straight line satisfying certain natural Diophantine conditions. The former implies that

$$\dim \left( \bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t; \boldsymbol{\theta}_t) \cap \mathcal{C} \right) = 1,$$

which strengthens even the homogeneous results obtained in [4, 5] that solve an old problem of Davenport. These winning results will be the subject of a forthcoming joint paper with Jinpeng An.

### 3.1 Proof of Theorem 4

Throughout, we fix  $i, j > 0$  satisfying (26) and  $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$ . The situation when either  $i = 0$  or  $j = 0$  is easier and will be omitted.

Since  $\mathbf{Bad}(i, j; \boldsymbol{\theta}) \subseteq \mathbb{R}^2$ , we obtain for free the upperbound result:

$$\dim \mathbf{Bad}(i, j; \boldsymbol{\theta}) \leq 2.$$

Thus, the proof reduces to establishing the complementary lowerbound. With this in mind, for a fixed constant  $c > 0$  let

$$\mathbf{Bad}_c(i, j; \boldsymbol{\theta}) := \{(x_1, x_2) \in \mathbb{R}^2 : \max\{\|qx_1 - \theta_1\|^{1/i}, \|qx_2 - \theta_2\|^{1/j}\} > c/q \quad \forall q \in \mathbb{N}\}.$$

Clearly  $\mathbf{Bad}_c(i, j; \boldsymbol{\theta}) \subset \mathbf{Bad}(i, j; \boldsymbol{\theta})$  and

$$\mathbf{Bad}(i, j; \boldsymbol{\theta}) = \bigcup_{c>0} \mathbf{Bad}_c(i, j; \boldsymbol{\theta}).$$

Geometrically, the set  $\mathbf{Bad}_c(i, j; \boldsymbol{\theta})$  consists of points  $(x_1, x_2) \in \mathbb{R}^2$  which avoid all rectangles centred at shifted rational points  $((\theta_1 - p_1)/q, (\theta_2 - p_2)/q)$  with sidelengths  $2c^i q^{-(1+i)}$



and  $2c^j q^{-(1+j)}$ . The sides are taken to be parallel to the coordinate axes. The overall strategy is to construct a ‘Cantor-type’ subset  $\mathbf{K}_c^\theta (= \mathbf{K}_c^\theta(i, j))$  of  $\mathbf{Bad}_c(i, j; \theta)$  with the property that  $\dim \mathbf{K}_c^\theta \rightarrow 2$  as  $c \rightarrow 0$ . This together with the fact that

$$\dim \mathbf{Bad}(i, j; \theta) \geq \dim \mathbf{Bad}_c(i, j; \theta) \geq \dim \mathbf{K}_c^\theta$$

implies the required lower bound result.

To obtain the desired Cantor type set  $\mathbf{K}_c^\theta$ , we adapt the homogeneous construction of  $\mathbf{K}_c = \mathbf{K}_c^0$  given in [29, §3.1] that is at the heart of establishing (27); that is to say Theorem 4 with  $\theta = 0$ .

### 3.1.1 The homogeneous construction

Let  $R \geq 11$  be an integer and  $c > 0$  be given by

$$c := 8^{-1/i} R^{-2(1+i)/i}. \quad (28)$$

It is established in [29, §3.1], by induction on  $n \geq 0$ , the existence of a nested collection  $\mathcal{F}_n$  of closed rectangles  $F_n := I_n \times J_n$  with the property that for all points  $(x_1, x_2) \in F_n$  the following (homogeneous) condition is satisfied:

$$\max\{ \|qx_1\|^{1/i}, \|qx_2\|^{1/j} \} > c q^{-1} \quad \forall \quad 0 < q < R^n. \quad (\text{H})$$

The lengths of the sides  $I_n$  and  $J_n$  of  $F_n$  are given by

$$|I_n| := \frac{1}{4} R^{-(1+i)(n+1)} \quad \text{and} \quad |J_n| := \frac{1}{4} R^{-(1+j)(n+1)} \quad (n \geq 0). \quad (29)$$

Without loss of generality assume that  $0 < i \leq j < 1$  so that the rectangles  $F_n$  are long and thin unless  $i = j$  in which case the rectangles are obviously squares.

The crux of the induction is as follows. We work within the closed unit square and start by subdividing the square into closed rectangles  $F_0$  of size  $I_0 \times J_0$  – starting from the bottom left hand corner of the unit square (i.e. the origin). Denote by  $\mathcal{F}_0$  the collection of rectangles  $F_0$ . For  $n = 0$ , condition (H) is trivially satisfied for any rectangle  $F_0 \in \mathcal{F}_0$ , since there are no integers  $q$  satisfying  $0 < q < 1$ . Given  $\mathcal{F}_n$  satisfying condition (H), we wish to construct a nested collection  $\mathcal{F}_{n+1}$  for which the condition is satisfied for  $n + 1$ . Suppose  $F_n$  is a good rectangle; that is, all points  $(x_1, x_2) \in F_n$  satisfy condition (H). In short,  $F_n \in \mathcal{F}_n$ . Now partition  $F_n$  into rectangles  $F_{n+1} := I_{n+1} \times J_{n+1}$  – starting from the bottom left hand corner of  $F_n$ . From (29), it follows that there are  $[R^{1+i}] \times [R^{1+j}]$  rectangles in the partition. Since they are nested, anyone of these rectangles will satisfy condition (H) for  $n + 1$  if for any point  $(x_1, x_2)$  in  $F_{n+1}$  the inequality

$$\max\{ \|qx_1\|^{1/i}, \|qx_2\|^{1/j} \} > c q^{-1} \quad (30)$$

is satisfied for

$$R^n \leq q < R^{n+1}. \quad (31)$$

With  $q$  in this ‘denominator’ range, suppose there exists a bad rational pair  $(p_1/q, p_2/q)$  so that (30) is violated, in other words

$$|x_1 - p_1/q| \leq c^i q^{-(1+i)} \quad \text{and} \quad |x_2 - p_2/q| \leq c^j q^{-(1+j)}$$

for some point in  $F_n$  and therefore in some  $F_{n+1}$ . Such  $F_{n+1}$  rectangles are bad in the sense that they do not satisfy condition (H) for  $n+1$  and those that remain are good. The upshot of the ‘Stage 1’ argument in [29, §3.1] is that there are at most

$$3 \left[ \frac{|J_n|}{|J_{n+1}|} \right] \leq 3R^{1+j} \quad (32)$$

bad  $F_{n+1}$  rectangles in  $F_n$ . Hence, out of the potential  $[R^{1+i}] \times [R^{1+j}]$  rectangles, at least

$$(R^{1+i} - 1)(R^{1+j} - 1) - 3R^{1+j} > R^3(1 - 5R^{-(1+i)})$$

are good  $F_{n+1}$  rectangles in  $F_n$ . Now choose exactly  $[R^3(1 - 5R^{-(1+i)})]$  of these good rectangles and denote this collection by  $\mathcal{F}(F_n)$ . Finally, define

$$\mathcal{F}_{n+1} := \bigcup_{F_n \subset \mathcal{F}_n} \mathcal{F}(F_n).$$

Thus, given the collection  $\mathcal{F}_n$  for which condition (H) is satisfied for  $n$ , we have constructed a nested collection  $\mathcal{F}_{n+1}$  for which condition (H) is satisfied for  $n+1$ . This completes the proof of the induction step and so the construction of the Cantor-type set

$$\mathbf{K}_c := \bigcap_{n=0}^{\infty} \mathcal{F}_n.$$

### 3.1.2 Bringing the inhomogeneous approximation into play

The idea is to merge the inhomogeneous approximation constraints into the above homogeneous construction. In short, this involves creating a subcollection  $\mathcal{F}_n^\theta$  of  $\mathcal{F}_n$  so that for all points  $(x_1, x_2) \in F_n$  with  $F_n \subseteq \mathcal{F}_n^\theta$ , both the (homogeneous) condition (H) and the following (inhomogeneous) condition are satisfied:

$$\max\{ \|qx_1 - \theta_1\|^{1/i}, \|qx_2 - \theta_2\|^{1/j} \} > c_* q^{-1} \quad \forall \quad 0 < q < R^{n-d}, \quad (\text{I})$$

where

$$c_*^i := \frac{1}{8} R^{-(1+i)(d+2)} \quad \text{and} \quad d := \left\lceil \frac{3}{i} \right\rceil. \quad (33)$$

For  $n = 0$ , condition (I) is trivially satisfied for any rectangle  $F_0 \subset \mathcal{F}_0$ , since there are no integers  $q$  satisfying  $0 < q < 1$ . Put  $\mathcal{F}_0^\theta := \mathcal{F}_0$ . Now suppose  $\mathcal{F}_n^\theta \subseteq \mathcal{F}_n$  has been constructed

and for each  $F_n$  in  $\mathcal{F}_n^\theta$  construct the collection  $\mathcal{F}(F_n)$  as before. Then by definition, each  $F_{n+1} \in \mathcal{F}(F_n)$  satisfies condition (H) for  $n+1$ . The aim is to construct a subcollection  $\mathcal{F}^\theta(F_n)$  such that for each  $F_{n+1}$  in  $\mathcal{F}^\theta(F_n)$  condition (I) for  $n+1$  is also satisfied; in other words, for any point  $(x_1, x_2)$  in  $F_{n+1}$  the inequality

$$\max\{ \|qx_1 - \theta_1\|^{1/i}, \|qx_2 - \theta_2\|^{1/j} \} > c_* q^{-1} \quad (34)$$

is satisfied for

$$R^{n-d} \leq q < R^{n+1-d}. \quad (35)$$

With  $q$  satisfying (35), suppose there exists a bad shifted rational pair  $((\theta_1 + p_1)/q, (\theta_2 + p_2)/q)$  so that (34) is violated, in other words

$$|x_1 - (\theta_1 + p_1)/q| \leq c_*^i q^{-(1+i)} \quad \text{and} \quad |x_2 - (\theta_2 + p_2)/q| \leq c_*^j q^{-(1+j)}$$

for some point  $(x_1, x_2)$  in  $F_n$ . Then, in view of (35), it follows that

$$|x_1 - (\theta_1 + p_1)/q| \leq c_*^i R^{-(1+i)(n-d)} \leq \frac{1}{2} |I_{n+1}| \quad (36)$$

if

$$c_*^i \leq \frac{1}{8} R^{-(1+i)(d+2)}. \quad (37)$$

Similarly,

$$|x_2 - (\theta_2 + p_2)/q| \leq \frac{1}{2} |J_{n+1}| \quad (38)$$

if

$$c_*^j \leq \frac{1}{8} R^{-(1+j)(d+2)}. \quad (39)$$

Observe that (37) implies (39). In view of (33), we have equality in (37), thus (36) and (38) are satisfied and it follows that any bad shifted rational pair gives rise to at most four bad rectangles  $F_{n+1}$  in  $\mathcal{F}(F_n)$ ; i.e. rectangles for which (I) is not satisfied for  $n+1$ . Now suppose there exist two bad shifted rational pairs, say  $((\theta_1 + p_1)/q, (\theta_2 + p_2)/q)$  and  $((\theta_1 + \tilde{p}_1)/\tilde{q}, (\theta_2 + \tilde{p}_2)/\tilde{q})$ . Then, for any  $(x_1, x_2) \in F_n$  we have that

$$\begin{aligned} |x_1 - (\theta_1 + p_1)/q| &\stackrel{(36)}{\leq} \frac{1}{2} |I_{n+1}| + |I_n| < 2|I_n| \\ \implies |qx_1 - \theta_1 - p_1| &< q 2|I_n| \stackrel{(35)}{\leq} 2R^{n+1-d}|I_n| \end{aligned}$$

and

$$\begin{aligned} |x_2 - (\theta_2 + p_2)/q| &\stackrel{(38)}{\leq} \frac{1}{2} |J_{n+1}| + |J_n| < 2|J_n| \\ \implies |qx_2 - \theta_2 - p_2| &< q 2|J_n| \stackrel{(35)}{\leq} 2R^{n+1-d}|J_n|. \end{aligned}$$

Similarly, we obtain that

$$|\tilde{q}x_1 - \theta_1 - \tilde{p}_1| < 2R^{n+1-d}|I_n| \quad \text{and} \quad |\tilde{q}x_2 - \theta_2 - \tilde{p}_2| < 2R^{n+1-d}|J_n|.$$

Let  $q_* := |q - \tilde{q}|$  and observe that

$$0 < q_* < R^{n+1-d} < R^n \quad (40)$$

It now follows that

$$\begin{aligned} |q_* x_1 - (p_1 + \tilde{p}_1)| &= |(qx_1 - \theta_1 - p_1) - (\tilde{q}x_1 - \theta_1 - \tilde{p}_1)| \\ &\leq |qx_1 - \theta_1 - p_1| + |\tilde{q}x_1 - \theta_1 - \tilde{p}_1| \\ &\leq 4R^{n+1-d}|I_n| \leq R^{-d(1+i)}q_*^{-i} \stackrel{(33)}{\leq} R^{-3(1+i)}q_*^{-i} \\ &\stackrel{(28)}{\leq} c^i q_*^{-i}. \end{aligned} \quad (41)$$

Similarly,

$$\begin{aligned} |q_* x_2 - (p_2 + \tilde{p}_2)| &= |(qx_2 - \theta_2 - p_2) - (\tilde{q}x_2 - \theta_2 - \tilde{p}_2)| \\ &\leq |qx_2 - \theta_2 - p_2| + |\tilde{q}x_2 - \theta_2 - \tilde{p}_2| \\ &\leq 4R^{n+1-d}|J_n| \leq R^{-d(1+j)}q_*^{-j} \stackrel{(33)}{\leq} R^{-\frac{(3-i)}{i}(1+j)}q_*^{-j} \\ &\stackrel{(28)}{\leq} c^j q_*^{-j}. \end{aligned} \quad (42)$$

The upshot of inequalities (40), (41) and (42) is that the homogeneous condition (H) is not satisfied for points in  $F_n$ . This contradicts the fact that  $F_n \in \mathcal{F}_n^\theta \subseteq \mathcal{F}_n$ . In turn, this implies that there exists at most one bad shifted rational pair that gives rise to at most 4 bad  $F_{n+1}$  rectangles amongst those in  $\mathcal{F}(F_n)$ . In other words, at least

$$\#\mathcal{F}(F_n) - 4 = [R^3(1 - 5R^{-(1+i)})] - 4 > R^3(1 - 6R^{-(1+i)})$$

of the  $F_{n+1}$  rectangles satisfy both conditions (H) and (I) for  $n+1$ . Now choose exactly  $[R^3(1 - 6R^{-(1+i)})]$  of these good rectangles and denote this collection by  $\mathcal{F}^\theta(F_n)$ . Finally, define

$$\mathcal{F}_{n+1}^\theta := \bigcup_{F_n \in \mathcal{F}_n^\theta} \mathcal{F}^\theta(F_n) \quad \text{and} \quad \mathbf{K}_c^\theta := \bigcap_{n=0}^{\infty} \mathcal{F}_n^\theta.$$

### 3.1.3 The finale

It remains to show that

$$\dim \mathbf{K}_c^\theta \rightarrow 2 \quad \text{as} \quad c \rightarrow 0.$$

This involves essentially following line by line the arguments set out in [29, §3.2 and §3.3]. The details are left to the reader.

□

**Acknowledgements.** SV would like to thank Dani for his support during the early years of his mathematical life – basically when it mattered most! Naturally, an enormous thanks to the delightful and magnificent Bridget, Ayesha and Iona. The younger two are curious and challenging for all the right reasons – long may you remain that way and good luck with high school! This brings me onto their wonderful teachers during the last two years at primary school who have instilled them with great belief and have been superb role models for both children and adults. For this and much much more, I would like to take this opportunity to thank Mr Youdan, Mr Middleton and Mrs Davison. I hope you continue to inspire!

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